Sean A. Hayward*
Asia Pacific Center for Theoretical Physics,
The Korea Foundation for Advanced Study Building 7th Floor, Yoksam-dong 678-39, Kangnam-gu, Seoul 135-081, Korea
and
Department of Physics, Konkuk University, 93-1 Mojin-dong, Kwangjin-gu, Seoul 143-701, Korea
(20th December 2000)

Gravitational waves are given a local definition in a quasi-spherical approximation, describing roughly spherical but otherwise dynamical astrophysical objects, such as a black hole forming by binary black-hole coalescence. A local effective energy tensor is defined for the gravitational waves, satisfying standard energy conditions. Radiation reaction, such as the back-reaction of the gravitational waves on the black hole, may then be described by including the gravitational-wave energy tensor as a source in the truncated Einstein equations. This can be formulated as a second quasi-spherical approximation, which retains non-linear terms in the fields encoding the gravitational waves. The energy-momentum in a canonical frame is covariantly conserved. The strain to be measured by a distant detector is simply defined.

04.30.-w, 04.25.-g, 04.20.Ha, 04.70.Bw

Gravitational-wave astronomy is expected to become a major observational science in the coming millenium. Gravitational-wave theory, however, is based on various approximations, as there is no general local definition of a gravitational wave in General Relativity; see e.g the review of Thorne [1]. Einstein [2] initiated the study of gravitational waves, in the linearized approximation on flat space-time, but there were objections to their reality and ability to carry energy. General belief came only in the 1960s with two further approximations. Firstly, assuming asymptotic flatness allowed a definition of gravitational wave infinitely far from all sources, as initiated by Bondi and formalized in the conformal framework of Penrose [3–9]. Secondly, the high-frequency or short-wavelength approximation allowed a locally averaged definition of gravitational wave, as formulated by Isaacson after initial ideas of Wheeler [10–14]. In both cases, the physical reality of the waves is indicated by definitions of their energy which, added to that of the matter, yields energy-balance equations: the Bondi energy-loss equation and the averaged Einstein equation, respectively. This letter describes how this may also be achieved in a quasi-spherical approximation scheme. Moreover, the resulting energy tensor of the waves is local. Thus the approximation allows a definition of gravitational wave at any point.

The recent quasi-spherical approximation [15] was intended to be applicable to coalescing black-hole binaries, one of the main expected sources for gravitational-wave detectors. It may also prove applicable to neutron stars or supernovas, or indeed any astrophysical situation which has rough spherical symmetry. Further, for any other process which can be enclosed by roughly spherical surfaces, it provides a mid-zone and, assuming isolation, far-zone approximation. Mathematically this is achieved by linearizing certain fields which would vanish in exact spherical symmetry, having made a decomposition of space-time adapted to the roughly spherical surfaces, henceforth called transverse surfaces. Since this makes no assumption of closeness to stationarity, the approximation holds for arbitrarily fast dynamical processes. The approximation has also been tested against angular momentum by applying it to Kerr black holes [16]: the error in the strain waveform is much lower than expected signals from binary black-hole coalescence.

The wavefronts of outgoing and ingoing gravitational waves form two families of null hypersurfaces, intersecting in the two-parameter family of transverse spatial surfaces. This geometry is described by the formalism of dual-null dynamics [17,18], briefly summarized here. Labelling the hypersurfaces by null coordinates x^+ and x^-, and taking coordinates x^a for the transverse surfaces, the space-time metric takes the form

$$g = h_{ab}(dx^a + s^a_+ dx^+ + s^a_- dx^-) \otimes (dx^b + s^b_+ dx^+ + s^b_- dx^-) - 2e^{-f} dx^+ \otimes dx^- \qquad (1)$$

where $\otimes$ denotes the symmetric tensor product, h is the metric of the transverse surfaces, $s_\pm$ are two shift vectors and f is a normalization function; where appropriate, Latin letters are used for transverse indices and Greek letters for space-time indices. The covariant derivative of h is denoted by D, and the evolution derivatives are

$$\Delta_\pm = \bot L_{\partial/\partial x^\pm} \qquad (2)$$

where L denotes the Lie derivative and $\bot$ denotes projection by h. It is also useful to decompose h into a conformal factor Ω and a conformal metric k by

$$h_{ab} = \Omega^{-2} k_{ab} \qquad (3)$$

such that the Hodge operator $\hat{*}$ of k is given by the standard spherical-polar area form

*hayward@mail.apctp.org

$$\hat{*}1 = \sin\theta\, d\theta \wedge d\phi \tag{4}$$

where $x^a = (\theta, \phi)$ are quasi-spherical polar coordinates and $\wedge$ denotes the exterior product of forms. Then Ω^{-1} is the quasi-spherical radius. The extrinsic curvature of the dual-null foliation may be encoded in conformally rescaled expansions $\vartheta_\pm$ and traceless shears $\varsigma_{\pm ab}$, inaffinities $\nu_\pm$ and twist ω_a, defined previously [15] and below as necessary. The Einstein equation and contracted Bianchi identity may then be written in a first-order form expressing $\Delta_\pm$ derivatives of the dynamical variables $(\Omega, k, f, s_\pm, \vartheta_\pm, \varsigma_\pm, \nu_\pm, \omega)$ in terms of the dynamical variables and their first and second transverse derivatives.

The *first quasi-spherical approximation*, introduced previously [15], consists of linearizing in $(\varsigma_\pm, s_\pm, \omega, D)$, which vanish in spherical symmetry [19,20]. This yields a greatly simplified set of truncated equations, decoupling into a three-level hierarchy. Moreover, the last level, the equations for $(s_\pm, \omega)$, need not be solved for the radiation problem. This is because, fixing Δ_+ to be the outgoing derivative, the Bondi news at null infinity $\Im^\pm$ is essentially $\varsigma_\mp = \Omega^{-1}\Delta_\mp k$ [15,16,21].[1] Specifically, the conformal strain tensor at future null infinity $\Im^+$ is

$$\varepsilon = \frac{1}{2}\int \varsigma_-\, dx^-. \tag{5}$$

This means that

$$\epsilon = \frac{\varepsilon}{R} \tag{6}$$

is the strain tensor at a large distance R from the source, so that the displacements to be measured by a gravitational-wave detector are

$$\frac{\delta\ell}{\ell} = \epsilon_{ab} e^a e^b \tag{7}$$

where the unit vector e is the direction of displacement. Thus the variables of the quasi-spherical approximation are directly related to the observable strain; no further far-zone approximation is required.

Moreover, the Bondi flux may be localized in terms of a gravitational-wave energy flux, derived below as the appropriate contraction of an effective *energy tensor of the gravitational waves*:

$$\Theta_{\alpha\beta} = \frac{\langle \Delta_\alpha k, \Delta_\beta k\rangle - \frac{1}{2} g^{\gamma\delta}\langle \Delta_\gamma k, \Delta_\delta k\rangle g_{\alpha\beta}}{32\pi} \tag{8}$$

where $\langle \alpha, \beta\rangle = k^{ab}k^{cd}\alpha_{ac}\beta_{bd}$, $\Delta_a = 0$ and units are such that Newton's gravitational constant is unity. Explaining this is the main purpose of this letter. Principally, Θ acts like a matter energy tensor in the truncated Einstein equations and satisfies a covariant conservation law, conservation of energy. It should be stressed that (i) mathematically, Θ is a genuine tensor, but depends on the dual-null foliation, not just on the space-time; (ii) the physical interpretation of Θ as energy requires the quasi-spherical approximation to be valid, meaning that the transverse surfaces must indeed be roughly spherical. This will not be made precise here, as the range of validity of the approximation is not clear in advance and best explored in applications. The intuitive meaning of roughly spherical should be clear by any standards.

In short, the quasi-spherical approximation allows a local definition of the energy-momentum-stress of gravitational waves, and therefore of a gravitational wave itself: a *gravitational wave* is present at a given point if and only if Θ is non-zero there. With the above orientation, there is an outgoing wave if and only if $\Delta_- k$ (equivalently ς_-) is non-zero, and an ingoing wave if and only if $\Delta_+ k$ (equivalently ς_+) is non-zero. As an energy tensor, Θ satisfies the strong, dominant, weak and null energy conditions [22]. Thus *gravitational waves carry positive energy*. The terminology "radiation" instead of "wave" might be preferred in general, since $\Delta_\pm k$ need not be oscillatory. Instead, frequency spectra for outgoing and ingoing waves may be defined by Fourier transformations of $\Delta_\pm k$ from $x^\pm$ (at constant $x^\mp$) to frequency $f_\pm$.

The gravitational-wave energy tensor of the Isaacson high-frequency approximation [13,14] has a similar form to Θ, except that the term proportional to g disappears in the former. In particular, writing the non-zero components

$$\Theta_{\pm\pm} = ||\Delta_\pm k||^2/32\pi \tag{9}$$
$$\Theta_{ab} = e^f \langle \Delta_+ k, \Delta_- k\rangle h_{ab}/32\pi \tag{10}$$

where $||\alpha||^2 = \langle \alpha, \alpha\rangle$, one sees that the $\Theta_{\pm\pm}$ components have a similar form, with k replacing the transverse traceless metric perturbation and no averaging over several wavelengths as required by the high-frequency approximation. Earlier attempts to construct pseudotensors for gravitational waves by Einstein and others might be converted to genuine tensors by gauge-fixing adapted to the transverse surfaces, but currently such pseudotensors are generally not accepted.

Apart from the fact that a gravitational wave generally has two polarizations, as encoded in the two independent components of $\Delta_\pm k$, Θ is analogous to the energy tensor of Einstein-Rosen gravitational waves [23], which takes the massless Klein-Gordon form in terms of a gravitational potential generalizing the Newtonian potential. In both cases, there is generally a transverse radiation pressure produced by a combination of ingoing and outgoing gravitational waves, as well as the expected radial radiation pressure. The fact that $\Theta_{+-} = 0$ may be interpreted as meaning that gravitational waves are purely radiative and workless, as for the massless (but not massive) Klein-Gordon field.

[1] Generally, one should use the null derivatives $\Delta_\pm - \perp L_{s_\pm}$; however, applied to transverse tensors, they coincide with $\Delta_\pm$ in the quasi-spherical truncation, so will not be distinguished in this article.

If Θ is to measure the effective energy of the gravitational waves, then *radiation reaction*, the back-reaction of the waves on the space-time, should be described by including Θ as a matter energy tensor in the truncated Einstein equations. In fact, there is a more logical way to formulate this, suggested by noting that the gravitational waves are encoded in the conformal shears $\varsigma_\pm$, which are linearized in the first approximation. Then retaining non-linear terms in $\varsigma_\pm$ should give a more accurate approximation for the gravitational-wave sector of the theory. Thus the *second quasi-spherical approximation,* to be described in detail in a longer article [24], consists of linearizing in $(s_\pm, \omega, D)$ only. The resulting equations also decouple, this time into only two levels, with the last level for $(s_\pm, \omega)$ again being irrelevant to the radiation problem. The remaining equations are, taking the vacuum case,

$$\Delta_\pm \Omega = -\tfrac{1}{2}\Omega^2 \vartheta_\pm \tag{11}$$
$$\Delta_\pm f = \nu_\pm \tag{12}$$
$$\Delta_\pm k = \Omega \varsigma_\pm \tag{13}$$
$$\Delta_\pm \vartheta_\pm = -\nu_\pm \vartheta_\pm - \tfrac{1}{4}\Omega ||\varsigma_\pm||^2 \tag{14}$$
$$\Delta_\pm \vartheta_\mp = -\Omega(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f}) \tag{15}$$
$$\Delta_\pm \nu_\mp = -\Omega^2(\tfrac{1}{2}\vartheta_+\vartheta_- + e^{-f} - \tfrac{1}{4}\langle \varsigma_+, \varsigma_- \rangle) \tag{16}$$
$$\Delta_\pm \varsigma_\mp = \Omega(\varsigma_+ \circ \varsigma_- - \tfrac{1}{2}\vartheta_\mp \varsigma_\pm) \tag{17}$$

where $(\alpha \circ \beta)_{ab} = k^{cd}\alpha_{ac}\beta_{bd}$. The first three equations effectively define $(\vartheta_\pm, \nu_\pm, \varsigma_\pm)$. The first approximation may, of course, be recovered by linearizing in $\varsigma_\pm$. The additional quadratic terms in $\varsigma_\pm$ appear in the same way that a matter energy tensor with the form of Θ would, as can be seen by comparing with the spherically symmetric equations [19,20]. More formally, one may introduce a truncated Einstein tensor C [24] in the first approximation. Then the truncated Einstein equations are $C = 8\pi T$ in the first approximation, where T is the energy tensor of the matter, and $C = 8\pi(T + \Theta)$ in the second approximation. This is the first reason for identifying Θ as an energy tensor for the gravitational waves: it plays the role of an effective matter energy tensor in the second approximation.

Mathematically, the difference between first and second approximations is that in the first approximation, the equations for $(\Omega, f, \vartheta_\pm, \nu_\pm)$, the variables which survive in spherical symmetry, decouple from the equations for $(k, \varsigma_\pm)$, which constitute a wave equation for k. Physically this describes gravitational-wave propagation on a quasi-spherical background. The background is not fixed in advance and need not be spherically symmetric, so even the first approximation is widely applicable. There is no such decoupling in the second approximation: the gravitational-wave terms $(k, \varsigma_\pm)$ now enter the equations for the quasi-spherical part of the geometry, which thus reacts to the passage of the waves. Radiation reaction has thereby been included; there is no longer a background which is independent of the waves.

Nevertheless, both first and second approximations share the remarkable feature that, to compute the observable waveforms, no transverse derivatives need be considered. The truncated equations form an effectively two-dimensional system, to be integrated independently at each angle of the sphere. Physically this means that the observed gravitational-wave signal depends only on the line of sight to the source, surely a plausible result. Moreover, by virtue of the dual-null formulation, the equations are already written in characteristic form, the mathematically standard form for analysis of hyperbolic equations. Numerical implementation is consequently straightforward and computationally inexpensive. Numerical codes exist for both first and second approximations [16].

The terminology "first and second order" has been carefully avoided because the true second-order quasi-spherical approximation would be full Einstein gravity, for which the only non-linear terms in the dynamical fields and operators are second-order. However, one may say that the second approximation is second-order in the gravitational waves. More formally, one could introduce two expansion parameters into the full Einstein equations, ε_1 preceding $\varsigma_\pm$ and ε_0 preceding $(s_\pm, \omega, D)$. Then both approximations ignore terms $o(\varepsilon_0)$, whereas the first approximation ignores terms $o(\varepsilon_1)$. Then ε_1 measures the strength of the gravitational waves, whereas ε_0 measures other asphericities, due to angular momentum or other transverse effects.

The second reason for identifying Θ as an energy tensor is that, added to the energy tensor of the matter, it yields a covariant energy conservation law. This is derived in general in a longer article [24] and in the vacuum case as follows, by a more direct method. As in spherical symmetry [19,20], there is a canonical flow of time defined by the vector or 1-form

$$\xi = *d\Omega^{-1} \tag{18}$$

where $*$ is the Hodge operator of the evolution space, $*1 = e^{-f} dx^+ \wedge dx^-$. Its non-zero components are

$$\xi_\pm = \mp \Delta_\pm \Omega^{-1}. \tag{19}$$

Then ξ is analogous to the Killing vector of a stationary space-time. The *energy-momentum density of the gravitational waves*, referred to the canonical flow, is the vector or 1-form

$$j_\alpha = -\Theta_{\alpha\beta}\xi^\beta \tag{20}$$

with non-zero components

$$j_\pm = \pm e^f \Theta_{\pm\pm} \Delta_\mp \Omega^{-1}. \tag{21}$$

This has the same form as the energy-momentum density of the matter in spherical symmetry [19,20]. The corresponding *energy flux of the gravitational waves* is the dual 1-form

$$\psi = *j. \tag{22}$$

The conformally rescaled flux

$$\varphi = \Omega^{-2}\psi \tag{23}$$

then has non-zero components

$$\varphi_\pm = -\frac{e^f \vartheta_\mp ||\varsigma_\pm||^2}{64\pi}. \tag{24}$$

These expressions have the same form as those for the Bondi flux at $\Im^\mp$ [15,16,21]. Thus the energy flux ψ, or more exactly the conformal flux φ, is a local generalization of the Bondi flux. Denoting the Hodge operator of g by $\star$, the quasi-spherical truncation identity

$$\star 1 = *1 \wedge \hat{*}\Omega^{-2} \tag{25}$$

allows the divergence of j to be written as

$$\nabla_\alpha j^\alpha = \star d \star j = \Omega^2 * d(\Omega^{-2} * j) = \Omega^2 * d\varphi. \tag{26}$$

The truncated Einstein equations (11–17) yield

$$\Delta_\pm \varphi_\mp = \frac{e^f \Omega}{64\pi}\left(\vartheta_+\vartheta_- \langle\varsigma_+, \varsigma_-\rangle + \tfrac{1}{4}||\varsigma_+||^2||\varsigma_-||^2\right) \tag{27}$$

and therefore

$$*d\varphi = e^f(\Delta_-\varphi_+ - \Delta_+\varphi_-) = 0. \tag{28}$$

Thus j is covariantly conserved:

$$\nabla_\alpha j^\alpha = 0. \tag{29}$$

Physically this represents *conservation of energy*, as in spherical symmetry [19,20]. In the presence of matter, it holds for the combined energy of the gravitational waves and matter [24], as in cylindrical symmetry [23]. The Noether charge associated with the Noether current j provides a definition of active gravitational mass-energy generalizing that of spherical symmetry, including the energy of the gravitational waves. Then energy conservation can be written in the form of a first law, as will be shown in the longer article [24].

In summary, the quasi-spherical approximation scheme allows local definitions of gravitational waves and their energy. The waves are superimposed on a background in the first approximation, but react back on the space-time in the second approximation. Comparing results of the first and second approximations, in a given situation, provides a valuable internal guide to their accuracy, independent of any other estimates. In particular, the results obtained for Kerr black holes are numerically indistinguishable in the first and second approximations [16].

Gravitational-wave theory has progressed since the review of Thorne [1] by higher-order post-Newtonian approximations [25] and close-limit approximations [26], which can be used to describe, respectively, the pre-coalescence and post-coalescence phases of a binary black-hole inspiral. The coalescence phase has long been thought to be tractable only by numerical methods, but despite great efforts [28], the desired waveforms are not yet known. The quasi-spherical approximation is intended to apply to the post-coalescence phase, thereby reducing the period for which full numerical simulations are required. Similarly, a quasi-equilibrium approximation has also recently been suggested for the later pre-coalescence phase [27].

Radiation reaction is an essential physical ingredient in the post-Newtonian and quasi-equilibrium approximations, but is not included in the close-limit approximation, or indeed any other perturbative approach with a fixed background. In comparison, the second quasi-spherical approximation allows a fully relativistic inclusion of radiation reaction for rapidly evolving black holes. It is applicable, unfortunately with as yet unknown accuracy, from the very moment of coalescence, meaning the appearance of trapped surfaces enclosing both original trapped regions. It may even provide estimates when applied some time before coalescence, in the region subsequently visible from $\Im^+$. The physical reason is that much of the information concerning the most violent phase of the coalescence will be lost inside the coalesced black hole.

As an illustration, the following scenario is plausible on physical grounds: a distorted black hole generally emits gravitational waves, which generally backscatter to produce ingoing waves, which are absorbed by the black hole, thereby increasing its mass and area and generally changing its shape, thereby also absorbing some outgoing radiation, thereby changing the profile of subsequent gravitational-wave emission, and so on by feedback. This entire process is actually described, step by step, by the truncated Einstein equations (11–17). The area-increase property was derived as the second law of black-hole dynamics [29], where a black hole is locally defined by a type of trapping horizon, which was also shown to be achronal and have topologically spherical sections. Corresponding spherically symmetric first [20] and zeroth [30] laws of black-hole dynamics admit quasi-spherical generalizations, to be presented in the longer article [24], involving a local definition of surface gravity and the above mass.

Thus we have an astrophysically realistic context in which both black holes and gravitational waves are locally defined, along with their physical attributes, with each influencing the other. Apart from practical applications to gravitational-wave astronomy, this provides a rich arena in which to advance theoretical understanding of the dynamical interaction between gravitational waves and black holes.

Acknowledgements. Thanks to Hisa-aki Shinkai and Jong Hyuk Yoon for discussions. Research partly supported by the APCTP visiting program and a research grant of Konkuk University.

[1] K S Thorne, in Three Hundred Years of Gravitation, ed. S W Hawking & W Israel (Cambridge University Press 1987).

[2] A Einstein, Preuss. Akad. Wiss. Berlin, Sitz. Phys.-Math. Kl. 688 (1916).

[3] H Bondi, Nature **179**, 1072 (1957).

[4] H Bondi, M G J van der Burg & A W K Metzner, Proc. Roy. Soc. Lond. **A269**, 21 (1962).

[5] R K Sachs, Proc. Roy. Soc. Lond. **A270**, 103 (1962).

[6] R K Sachs, in Relativity, Groups and Topology, ed. C DeWitt & B DeWitt (Gordon & Breach 1963).

[7] R Penrose, Phys. Rev. Lett. **10**, 66 (1963).

[8] R Penrose, in Relativity, Groups and Topology, ed. C DeWitt & B DeWitt (Gordon & Breach 1963).

[9] R Penrose & W Rindler, Spinors and Space-Time Vol. 2 (Cambridge University Press 1988).

[10] J A Wheeler, Phys. Rev. **97**, 511 (1955).

[11] E A Power & J A Wheeler, Rev. Mod. Phys. **29**, 480 (1957).

[12] D R Brill & J B Hartle, Phys. Rev. **135**, B271 (1964).

[13] R A Isaacson, Phys. Rev. **166**, 1263 & 1272 (1968).

[14] C W Misner, K S Thorne & J A Wheeler, Gravitation (Freeman 1973).

[15] S A Hayward, Phys. Rev. **D61**, 101503 (2000).

[16] H Shinkai & S A Hayward, Quasi-spherical approximation for rotating black holes (gr-qc/0008075, in revision).

[17] S A Hayward, Ann. Inst. H. Poincaré **59**, 399 (1993).

[18] S A Hayward, Class. Quantum Grav. **10**, 779 (1993).

[19] S A Hayward, Phys. Rev. **D53**, 1938 (1996).

[20] S A Hayward, Class. Quantum Grav. **15**, 3147 (1998).

[21] S A Hayward, Class. Quantum Grav. **11**, 3037 (1994).

[22] S W Hawking & G F R Ellis, The Large Scale Structure of Space-Time (Cambridge University Press 1973).

[23] S A Hayward, Class. Quantum Grav. **17**, 1749 (2000).

[24] S A Hayward, Gravitational waves and black holes in a second quasi-spherical approximation (in preparation).

[25] L Blanchet, in Einstein's Field Equations and their Physical Implications, ed. B Schmidt (Springer).

[26] J Pullin, Prog. Theor. Phys. Suppl. 136, 107 (1999).

[27] M D Duez, T Baumgarte & S L Shapiro, Computing the complete gravitational wavetrain from relativistic binary inspiral (gr-qc/0009064).

[28] Binary Black Hole Grand Challenge Alliance, Phys. Rev. Lett. **80**, 2512 (1998).

[29] S A Hayward, Phys. Rev. **D49**, 6467 (1994).

[30] S A Hayward, Black holes: new horizons (gr-qc/0008-071); to appear in Proceedings of the Ninth Marcel Grossmann Meeting (World Scientific).